“Easy” Representations and the QSF property for groups

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Abstract

We define the class of “easy” groups as the class of those groups \( \Gamma \) admitting an (inverse) representation (which is a map \( \rightarrow \Gamma \) satisfying several topological properties) for which the set of double points is closed. Our main result is that easy groups are QSF (i.e. quasi-simple filtrated).

Keywords: Discrete groups, quasi-simple filtration, geometric simple connectivity, inverse representations of groups.

MSC Subject: 57 M 05, 57 M 10, 57 N 35.

1 Introduction and definitions

In this paper we will deal only with finitely presented groups \( \Gamma \). For our purpose a very special kind of presentations will be used. To each finitely presented group \( \Gamma = \langle S | R \rangle \) with a finite set of generators \( S \) and a finite set of relators \( R \), one can associate a compact SINGULAR 3-manifold \( M^3(\Gamma) \) by the following kind of procedure.

Start with a smooth 3-dimensional handlebody \( H \) of index \( g \) corresponding to the generators. Then to each relator we will associate a curve \( S^1 \subset \partial H \). More explicitly, we consider a smooth generic immersion

\[
(1 - 1) \quad \sum_{i=0}^{r} S^1_i \xrightarrow{\alpha} \partial H
\]

for which we will take the immersed regular neighborhood

\[
(1 - 2) \quad \sum_{i=0}^{r} S^1_i \times [-\epsilon, \epsilon] \xrightarrow{\alpha} \partial H.
\]

The \( M^3(\Gamma) \) is gotten by adding \( r \) handles of index two to \( H \), along \((1 - 2)\). Without any loss of generality, after possibly enlarging \( g \) and \( r \), one may assume here, if one wishes to do so,

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that each individual $S^1_i$ is embedded. Anyway $M^3(\Gamma)$ has singular points when it fails to be a manifold. The singular set

$$\text{Sing } M^3(\Gamma) \subset M^3(\Gamma)$$

is a disjoint union of little squares $S \subset \partial H$, the connected components of the image of the set of double points of $(1 - 2)$ in $\partial H$, and these kind of singularities, we will call \textit{immortal} so as to distinguish them from the mortal singularities to appear later. In the neighborhood of each $x \in \text{int } S$, our $M^3(\Gamma)$ can be described as a

$$\{\text{figure} \} \times \mathbb{R}^2,$$

or alternatively as

$$\begin{aligned}
\text{(1 - 3) The union of three copies of the upper half-space } \mathbb{R}^3_+ \text{ along their common boundary.}

\text{The central notion of this paper will be the representations of } \Gamma. \text{ Contrary to the standard terminology, whereby “representations” mean morphisms } \Gamma \rightarrow \cdots, \text{ our representation will be arrows of the form } \cdots \rightarrow \widetilde{M}^3(\Gamma) = \{ \text{The universal covering space of } M^3(\Gamma) \}, \text{ an object which, up to quasi-isometry, is the same thing as } \Gamma \text{ itself. Such representations were already defined in [15, 16, 7], but since we want this paper to be, as much as it is possible, independently readable, we will review here the basics necessary.}

\text{Consider first some (non necessarily locally finite) simplicial complex of dimension } \leq 3 \text{ and a non-degenerate (simplicial) map}

$$X \xrightarrow{F} Y$$

\text{where } Y \text{ may be } M^3(\Gamma) \text{ or } \widetilde{M}^3(\Gamma) \text{ or even a smooth manifold } M^3.

\text{We will call (mortal) singularities of } F, \text{ the points } x \in X \text{ in the neighborhood of which } F \text{ fails to be immersive. Their set is denoted } \text{Sing}(F) \subset X.

\text{On the set } X, \text{ two equivalence relations } \Psi(F) \subset \Phi(F) \subset X \times X \text{ will be considered, namely}

(i) $\Phi(F)$ which is the set of pairs $(x, y)$ with $F(x) = F(y)$

(ii) $\Psi(F)$ is the smallest equivalence relation compatible with $F$, killing all the mortal singularities, so that the induced map $X/\Psi(F) \rightarrow Y$ is an immersion.

\text{It is shown in [8, 16] that this definition makes sense and we will not pursue this issue longer here. In the specific situation considered later in this paper, things will anyway be very concrete and explicit.}

\text{Definition 1.1} \text{ A 2-dimensional representation for } \Gamma \text{ is a non-degenerate simplicial map}

$$X^2 \xrightarrow{F} \widetilde{M}^3(\Gamma)$$

\text{with the following features}

\text{(1-A)} \text{ $X^2$ is GSC (i.e. geometrically simply connected), and what this means will be soon recalled.}
(1-B) \( \Psi(f) = \Phi(f) \); and in such a case one says that the representation \((1 - 5)\) is \textbf{zippable}.

(1-C) \( f \) is “essentially” surjective, in the sense that one can get \( \widetilde{M}^3(\Gamma) \) from \( \bar{f}X^2 \subset \widetilde{M}^3(\Gamma) \) by adding cells of dimension (or index) 2 and 3.

Before coming back to GSC, there is also a similar notion of 3\textsuperscript{d} representation, where condition (1-C) above is replaced by \( \Im f = \widetilde{M}^3(\Gamma) \). There is a fairly easy argument (see [16]) producing 3\textsuperscript{d} representations for any group \( \Gamma \) and it is not hard to extract 2\textsuperscript{d} representations from these. But all such low-cost representations are highly pathological and hence not of much use.

Now the notion GSC is well-known in differential topology, where it means the existence of a handlebody decomposition without handles of index \( \lambda = 1 \). But it also makes perfect sense in the context of cell-complexes. We are interested here in the non-compact situation when cells of dimension \( \lambda = 1, \lambda = 2, \ldots \) are attached to some infinite tree \( T \). Call them \( H_1^1, H_2^2, H_3^3, \ldots \).

With this, the cell complex \( Z \) will be said to be GSC if it possesses a cell decomposition
\[
T + \{1\text{-cells } H_1^1\} + \{2\text{-cells } H_2^2\} + \ldots
\]
where the 1-cells and 2-cells are in cancelling position. Explicitly, this means that for the family of \( H_1^1 \)'s one can find a subfamily \( \{H_2^2\} \subset \{ \text{all } H_2^2 \text{'s} \} \) such that the \textit{geometric intersection matrix}, counting how many times \( H_1^1 \) occurs in \( \partial H_2^2 \), has the easy \textit{id + nilpotent} form below
\[
\partial H_2^2 \cdot H_1^1 = \delta_{ij} + a_{ij}, \quad \text{where } a_{ij} \in \mathbb{Z}_+ \text{ is such that } a_{ij} > 0 \Rightarrow j > i.
\]

As a side remark, one can also consider the difficult \textit{difficult id + nil} by changing \( j > i \) to \( i > j \) in (1 - 7). This certainly does not mean handle cancellation (in the infinite case, the only one of concern here). The classical Whitehead manifold \( Wh^3 \), which is certainly not GSC, exhibits this difficult \textit{id + nil} feature. This ends our discussion of representations.

We remind now the reader the following notion, due to S. Brick and M. Mihalik (see [1, 17]). We are now in the simplicial category, and a locally compact space \( X \) is called QSF (i.e. \textbf{quasi-simple filtered}) if for every compact \( k \subset X \) there is another abstract simply connected compact \( K \), endowed with an inclusion \( i \) from \( k \), coming with a continuous (simplicial) map \( f \)
\[
\begin{array}{ccc}
k & \xleftarrow{i} & K \\
& \searrow & \\
X & \xrightarrow{f} & \end{array}
\]
exhibiting the Dehn-type property \( M_2(f) \cap i(k) = \emptyset \).

\textbf{Notations.} For any map \( A \xrightarrow{h} B \) we denote \( M_2(h) = \{ x \in A \text{ such that } \text{card } h^{-1}(x) < 1 \} \) while \( M_2(h) \subset A \times A \) is the set of pairs \( (x, y) \), with \( x \neq y \), such that \( h(x) = h(y) \).

We will not discuss here more on this notion for which there is a vast literature [1, 3, 17]. It suffices to stress the following three facts:

(I) The notion \( \Gamma \in \text{QSF} \) makes sense; in particular if \( \widetilde{M}^3(\Gamma) \in \text{QSF} \) then \( \Gamma \in \text{QSF} \) (and conversely).
(II) There is a trivial implication \( X \in \text{GSC} \Rightarrow X \in \text{QSF} \).

(III) The first author (D.O.) in collaboration with Louis Funar have proved the following in [3]: “We have \( \Gamma \in \text{QSF} \) \text{ I.F. AND ONLY I.F.} \text{ there is a smooth compact manifold} \( M \) such that \( \pi_1 M = \Gamma \) and \( M \in \text{GSC} \).

Definition 1.2 A \( 2^d \) representation (1 - 5) is called \textit{easy} if \( \text{Im}(f) = fX^2 \subset \tilde{M}^3(\Gamma) \) and \( M_2(f) \subset X^2 \) are \textit{closed} subsets.

Before discussing this notion a bit further, we will state the main result of this paper.

Theorem 1.3 \textit{If} \( \Gamma \) \textit{admits an easy representation, then} \( \Gamma \in \text{QSF} \).

We will actually prove a slightly stronger result, suggested to us by L. Funar. More generally then, in definition 1 before, consider \textit{QSF-representations} where (1 - A) is replaced by the weaker condition \( X^2 \in \text{QSF} \).

Theorem 1.4 \textit{If} \( \Gamma \) \textit{admits an easy QSF-representation, then} \( \Gamma \in \text{QSF} \).

Actually, the second author (V.P.) has developed a program for showing that all finitely presented group \( \Gamma \) are QSF. An outline [15] and a first part of the program [16] are available, while two more papers are waiting to be typed. Theorem 1 can be seen as an “easy”-model for what is going on in the full program and serve as a good introduction for it. Only few words about it here. To begin with, the reason for disguising \( \Gamma \) as a (singular) 3-manifold \( M^3(\Gamma) \) was to be able to make use of the richness of the double points set for mappings \( \dim 2 \rightarrow \dim 3 \) (see here the discussion in [7]). In real life one works with representations about which one can no longer assume that they are easy. But then, in the context of definition 1, it makes sense to look for an \( X^2 \) admitting a free \( \Gamma \)-action with equivariant \( f \). So, the first step is to have an equivariant representation (see [16]) with locally finite \( X^2 \). The next step is to control the zipping length: the (1 - B) in definition 1 implies that every \( (x, y) \in M^2(f) \) can be connected to \( \text{Sing}(f) \) via a continuous path \( \lambda(x, y) \) in \( M^2(f) \cup \text{Sing}(f) \). With some work one can achieve an (1 - 5) which is both equivariant and for which the length \( ||\lambda(x, y)|| \) (which is well-defined up to quasi-isometry) is uniformly bounded; actually \( \inf ||\lambda(x, y)|| \leq C \), for all \( (x, y) \in M^2(f) \). After that, the real work starts and one has to face the accumulation of \( M_2(f) \subset X^2 \) (of which the accumulation of \( \text{Im} f \subset \tilde{M}^3(\Gamma) \) is a sort of reflex).

We will not discuss these matters further here. Hopefully, the papers with full details should be soon available.

Concerning easy presentations, here are some facts:

A) All the hyperbolic groups of Gromov admit easy representations. This can be read between the lines in papers like [9, 10].

B) Using the full Thurston geometrization, proved by G.Perelman, one should certainly be able to show that all \( \pi_1 M^3 \)'s admit easy representations.

C) Nobody has actually ever seen a group which does not admit one. The second author conjectures that the following holds:

Conjecture 1 \textit{All finitely presented groups} \( \Gamma \) \textit{admit easy representations}.  

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But the proof of this should use, apart from some new ingredients, the full proof (and
results) of the second author’s theorem “All $\Gamma$ are QSF”:

This also means that even if we eventually can get rid of the accumulation of $M_2(f)$, one
has to start by fully living with it.

2 Proof of the “Easy Theorem”

We start with an easy QSF representation (1) for our group $\Gamma$ and notice that, since $M_2(f)$
and $fX^2$ are closed subsets, the $fX^2$ is a locally finite simplicial complex such that

$$\widetilde{M}^3(\Gamma) = fX^2 + \{ \text{cells of dimension } \lambda = 2 \text{ and } \lambda = 3 \}$$

Claim A: We can factor the map $f : X^2 \rightarrow fX^2$ as a sequence of elementary zipping
moves

$$X^2 = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots fX^2 \subset \widetilde{M}^3(\Gamma)$$

each of which is either

(3-1) to begin with, an elementary move $O(0), O(1), O(2)$ like in the figures 2,3,4. These moves
$O(i \leq 2)$ are “acyclic”, in the sense that they are homotopy equivalences. Because $\widetilde{M}^3(\Gamma)$
is singular, we have to add to the $O(i \leq 2)$ another acyclic move occurring when the
zipping hits $S$ and has to stop, like in (4-B) below. This movement “$O(S)$” changes a
mortal singularity into an immortal one. The $X_{i \geq 1}$’s can have such.

or (3-2) movements $O(3)$ like in the figure 5 which are, homotopically speaking, additions of 2-
cells.

Something very similar to this claim A is done in the papers [11, 12] by the second author
(V.P.) and in the review of that work by D. Gabai [4], in a context when $\widetilde{M}^3(\Gamma)$ is replaced by
a smooth homotopy 3-sphere $\Sigma^3$ and $X^2$ is a finite complex.

But something similar can easily be done in the present context, with two caveats. Firstly,
$\widetilde{M}^3(\Gamma)$ has now singularities and that issue will be discussed below. Secondly, we are now in
an infinite, non-compact context. But that should not make any problem.
An undrawable singularity $\sigma \in \text{Sing}(f)$. We have here the following local model: there is a smooth coordinate neighborhood $R^3 = \{(x, y, z)\} \subset M^3(\Gamma)$ with $f(\sigma) = (0, 0, 0)$. The open set $f^{-1}R^3 \subset X^2$ consists of the two planar “branches” $P_1, P_2$, glued at the source along the half-line $\frac{1}{2}L$ (ending at $\sigma$), and injected by $f$, transversally through each other. When double lines are present this is a mortal singularity $\sigma \in X^2$. But then, when no double points are present, this could also be an immortal singularity $\sigma \in \text{Sing}(fX^2)$, where $fX^2$ fails to embed in a 3-manifold, keeping $P_1, P_2$ transversal along $\frac{1}{2}L$, i.e. compatibly with $f$. The $s(n)$ will be discussed later.

Movement $O(0)$ at the source. At the end of the move there is no triple point, but a new mortal singularity at $\sigma' = \sigma''$. The new map has a lower card $M^3(f)$. 

Figure 1. An undrawable singularity $\sigma \in \text{Sing}(f)$. We have here the following local model: there is a smooth coordinate neighborhood $R^3 = \{(x, y, z)\} \subset M^3(\Gamma)$ with $f(\sigma) = (0, 0, 0)$. The open set $f^{-1}R^3 \subset X^2$ consists of the two planar “branches” $P_1, P_2$, glued at the source along the half-line $\frac{1}{2}L$ (ending at $\sigma$), and injected by $f$, transversally through each other. When double lines are present this is a mortal singularity $\sigma \in X^2$. But then, when no double points are present, this could also be an immortal singularity $\sigma \in \text{Sing}(fX^2)$, where $fX^2$ fails to embed in a 3-manifold, keeping $P_1, P_2$ transversal along $\frac{1}{2}L$, i.e. compatibly with $f$. The $s(n)$ will be discussed later.

Figure 2. Movement $O(0)$ at the source. At the end of the move there is no triple point, but a new mortal singularity at $\sigma' = \sigma''$. The new map has a lower card $M^3(f)$. 

Figure 2. Movement $O(0)$ at the source. At the end of the move there is no triple point, but a new mortal singularity at $\sigma' = \sigma''$. The new map has a lower card $M^3(f)$.
Figure 3: Movement $O(1)$. The figure is at the source, while below it is at the target. There can be any number of branches at $P$. Never mind, for the time being the letters $s(n)$ smeared on $X_i$, they will be explained later.

Figure 4: Movement $O(2)$. Here the mortal singularity dyes.
Figure 5. Movement O(3) where the mortal singularities $S'$ and $S''$ enter in a frontal collision and kill each other. The letters “s”, and “n” smeared in their neighborhoods on the respective branches $C$, $D$, correspond to some abstract desingularization at level $X_i$. As written here, the $O(3)$ movement is COHERENT. If, let us say at $S''$, we change $s(n)$ into $n(s)$ on $C$ and hence $n(s)$ into $s(n)$ on $D$, then we get the non-COHERENT movement $O(3)$.

A more pedantic way of describing (3) is the following. The $f$ defines the equivalence relation $\Phi(f) \subset X^2 \times X^2$ and so does every finite piece $X \rightarrow X_i$ define a similar $\Phi_i \subset X^2 \times X^2$, with $\Phi_i \subset \Phi_{i+1}$. The factorization in (3) simply means that $\Phi(f) = \cup_i \Phi_i$.

It should be noted that the various $X_i, i \geq 1$, are not necessarily QSF (certainly not GSC in the case of a usual, GSC representation), and so the $X_i \rightarrow \tilde{M}^3(\Gamma), i \geq 1$, are NOT representations, in the sense defined before.

So let us consider now three half spaces $U_1, U_2, U_3$ of $\tilde{M}^3(\Gamma)$ having in common exactly an immortal singularity $S = \partial U_1 \cap \partial U_2 \cap \partial U_3$. Any double line in $fM_2(f)$, starting let us say in $U_1$ and reaching at $S$, either continue transversally through $S$ or stops short. This leads to the following two possible local models.

(4 – A) There are two smooth sheets $A, B \subset X^2$ homeomorphic to $\mathbb{R}^2$ such that $f$ injects them inside $U_1 \cup U_2$ where $fA, fB, S$ are three planes in general position (coordinate planes).

Here the zipping proceeds through $S$, without paying any attention to $U_3$.

(4-B) $f$ injects $A$ and $B$ into $U_1 \cup U_2, U_1 \cup U_3$ respectively, and the zipping stops at $S$. Here $S$ generates an immortal singularity for $fX^2$. There are no mortal singularities for $fX^2$.

We will state now the next CLAIM, the proof of which will be given later on.
Claim B: For each dimension $n \geq 5$ we can choose CANONICALLY a smooth $n$-dimensional regular neighborhood, for each $X_{i \geq 0}$ in (3),

$$\Theta^n(X_i) \in \text{QSF}$$

such that the following things should happen:

6. The sequence of quotient space projections (3) can be changed into a sequence of smooth embeddings

$$(S_\infty) \quad \Theta^n(X^2) = \Theta^n(X_0) \subset \Theta^n(X_1) \subset \Theta^n(X_2) \subset \cdots$$

where:

6-a) If $X_l \rightarrow X_{l+1}$ is an acyclic $O(i)$ move, then $\Theta^n(X_l) \hookrightarrow \Theta^n(X_{l+1})$ is a compact, smooth Whitehead dilatation.

6-b) If $X_l \rightarrow X_{l+1}$ is an $O(3)$ move, then $\Theta^n(X_l) \hookrightarrow \Theta^n(X_{l+1})$ is the addition of a handle of index $\lambda = 2$.

Remark 2.1

1. If $\Theta^n$ would not be canonical, there would not be a priori any connection between $\Theta^n(X_l)$ and $\Theta^n(X_{l+1})$, giving rise to the embeddings above. But then, there is also a second use of canonicity. The $\Theta^n(M^3(\Gamma))$ and $\Theta^n(\tilde{M}^3(\Gamma))$ will also make sense (this is actually a claim too, part of CLAIM B, let us say) and canonicity will imply then that

$$\Theta^n(M^3(\Gamma)) \simeq \Theta^n(\tilde{M}^3(\Gamma))$$

2. Since $X_0$ is QSF, we automatically have $\Theta^n(X_0) \in \text{QSF}$. Next, the two embeddings $j_l$ from (6-a), (6-b) preserve the QSF, hence all $\Theta^n(X_i)$'s are QSF.

The final argument. Since $M_2(f) \subset X^2$ is a closed subset, any little open subset $U \subset \tilde{M}^3(\Gamma)$ can meet only finitely many elementary zipping moves from (3). It follows that we can put together the infinite sequence of embeddings $(S_\infty)$ into a smooth non-compact $n$-manifold with (large) non-empty boundary $\cup_{i=0}^\infty \Theta^n(X_i)$. Here, for any local piece of $\Theta^n(X_i)$ and any $i < j$, only finitely many of the intersections $\Theta^n(X_i) \cap (\Theta^n(X_{j+1}) - \Theta^n(X_j))$ are $\neq \emptyset$.

Since for every compact $k \subset \cup_{i=0}^\infty \Theta^n(X_i)$ there is an $i$ such that $k \subset \Theta^n(X_i) \subset \text{QSF}$, it follows that $\cup_{i=0}^\infty \Theta^n(X_i) \subset \text{QSF}$. Again, because of $M_2(f) \subset X^2$ being closed, we also have, inside $\Theta^n(\tilde{M}^3(\Gamma))$, the equality of sets

$$\cup_{i=0}^\infty \Theta^n(X_i) = \Theta^n(fX^2).$$

Because $fX^2 \subset \tilde{M}^3(\Gamma)$ is closed, so is $\Theta^n(fX^2) \subset \Theta^n(\tilde{M}^3(\Gamma))$. All this, together with (1 − C), implies that

$$\Theta^n(\tilde{M}^3(\Gamma)) = \Theta^n(fX^2) + \{\text{handles of ondex } \lambda = 2, \lambda = 3\}.$$ 

Hence $\Theta^n(\tilde{M}^3(\Gamma))$ is QSF, which, together with (7), implies that $\Gamma$ is QSF.
Lemma 1 Let $V^n$ be a smooth non-compact manifold with boundary $\partial V \neq \emptyset$ and let also $N^n = V^n + \{a \text{ handle } H^\lambda, \text{ with } \lambda > 1\}$.

If $V^n \in QSF$ then $N^n$ is also QSF.

Proof. Start with a compact $k \subset N^n$ and let $k_1 = k - H^\lambda \subset V^n$. Then go to $\partial H^\lambda = S^{\lambda-1} \times B^n \subset \partial V^n$, the attaching zone, and apply the QSF of $V^n$ for the compact $k_2 = k_1 \cup \partial H^\lambda \subset V^n$

$$\begin{array}{ccc}
k_2 & \xrightarrow{j} & K_2 \\
& \downarrow{f} & \\
V^n & \end{array}$$

We have that $\partial H^\lambda \subset K_2 - M_2(j)$, the compact $K = K_2 \cup H^\lambda$ is simply connected, contains $k$ and comes with a map into $N$ having the Dehn-property. This ends the proof. \hfill \Box

Remark 2.2

In this easy-version of the general statement “any $\Gamma \in QSF$”, we do not make use neither of equivariant representations nor on bounds on the zipping length, unlike what happens in [15, 16].

To complete the proof, it remains to show how one constructs the $\Theta^n(\ldots)$, with all the features above. Here we will borrow very heavily on the very initial, very easy part of the technology which the second author (V.P.) has developed in his approach to the Poincaré Conjecture (see [11, 12, 4]). But everything necessary for our present aim is explained here, and this paper is, at this point, totally self-contained.

We start by considering any map

$$(9) \quad Y^2 \xrightarrow{f} \tilde{M}^3(\Gamma)$$

where

(9-1) $Y^2$ is a locally finite simplicial complex

(9-2) the $\sigma \in \text{Sing}(f) \subset Y^2$, i.e. the non-immersive points, are like in figure 1

(9-3) we have to assume now that $Y^2$ has both mortal singularities $\text{Sing}(f)$ and immortal singularities $\text{Sing}(Y^2)$, disjoined from each other. The $\text{Sing}(f)$ are like the ones of (1), while the ones in $\text{Sing}(Y^2)$, which are also undrawable, are such that $f(\text{Sing}(Y^2)) \subset \text{Sing} \tilde{M}^3(\Gamma)$, and locally $f$ is here injective. So, for $y \in \text{Sing}(Y^2)$ we assume $f$ to be locally an injection, and the local figure is again the 1, but without any $M_2(f)$ present this time.

We will define now (abstract) desingularizations for our (9). By definition, such an abstract desingularization is a map

$$(10) \quad \{\text{The set of branches } P_1, P_2 \text{ (see figure 1) for each } \sigma \in \text{Sing}(f) + \text{Sing}(Y^2)\} \xrightarrow{\phi} \{S, N\}$$
(where \( \{S, N\} \) is the alphabet with two letters \( S, N \), such that for each individual \( \sigma \) and its \( P_1 = P_1(\sigma), P_2 = P_2(\sigma) \), we should have

\[
\phi(P_1(\sigma)) \neq \phi(P_2(\sigma)).
\]

The branch \( P \) coming with \( \phi = S \) will be called specified, the other one non-specified. This is illustrated in figure 1.

We consider now an elementary zipping move \( O(i \leq 3) \) or \( O(S) \), like the ones occurring in \( (S_\infty) \).

\[
\begin{array}{c}
Y^2 \\
\downarrow f \quad \downarrow f_1 \\
\quad M^3(\Gamma)
\end{array} \rightarrow \begin{array}{c}
Y_1^2 \\
\end{array}
\]

(11)

**Lemma 2** Any given abstract desingularization \( \phi \) for \( (Y^2, f) \) propagates canonically into an abstract desingularization \( \phi_1 \) for \( (Y_1^2, f_1) \).

**Proof.** The case \( O(1) \) is illustrated in figure 3 and the case \( O(0) \) from figure 2 can be treated similarly. In the cases \( O(2), O(3) \) one simply keeps by decree \( \phi_1 = \phi \) for the singularities not killed by the local move, and by decree too, one ignores the killed ones. The case \( O(S) \) should be obvious. \( \Box \)

**Remark 2.3**

1. In the context of \( (3) \), consider some arbitrarily given desingularization \( \phi \) at level \( X^2 = X_0 \), where \( \text{Sing} \ X_0 = \emptyset \). According to lemma 2, this propagates canonically through the whole of \( (3) \) inducing at each stage a desingularization which we continue to denote \( \phi \), for all \( X_i \)'s including \( X_\omega \overset{\text{def}}{=} f X^2 \).

At the final level \( \text{Sing} \ X_\omega \rightarrow \tilde{M^3}(\Gamma) = \emptyset \), while \( \text{Sing} X_\omega \nsubseteq X_\omega \cap \text{Sing} \tilde{M^3}(\Gamma) \neq \emptyset \), generically speaking. At intermediary levels we find both \( \text{Sing} \ X_i \rightarrow \tilde{M^3}(\Gamma) \neq \emptyset \) (mortal case) and \( \text{Sing} X_i \neq \emptyset \) (immortal case).

2. When the propagation of \( \phi \) from \( X_\omega \) to \( X_\omega \) is considered, the \( O(3) \)'s (and \( O(2) \)'s) are not in the way, we can always redirect the zipping flow so that it reaches to \( \text{Sing} X_\omega \) before performing \( O(3) \) (and \( O(2) \)). This is suggested in the next drawing.
So, without any loss of generality, the zipping strategy (3) is such that the $O(3)$’s are corralled at the very end.

To any abstract desingularization $\phi$ for $(Y^2, f_1)$ there is a canonically attached geometric desingularization

\[
\begin{array}{c}
\hat{Y}^2 = \tilde{Y}^2(\phi) \\
\downarrow \pi(\phi) \\
Y^2
\end{array}
\]

with the following feature:

(12-1) Any singularity $\sigma \in \text{Sing}(f) + \text{Sing}Y^2$ is blown up into a circle inside that local branch $P_1$ or $P_2$ (let us suppose it is in $P_1$), which is coming with $\phi = S$. This is suggested in the figure 7 below.
Notice that $\tilde{Y}^2(\phi)$ in (12) has a canonical smooth 3-dimensional regular neighborhood which comes with an immersion into $\overline{M}^3(\Gamma)$, guided by $f$ in (9); we will denote it by $\Theta^3(\tilde{Y}^2(\phi))$. Together with the abstract desingularization (10) comes not only the (12) but also a $\phi$-dependant 4-dimensional smooth regular neighborhood $\Theta^4(Y^2, \phi)$, together with a smooth embedding

\[ (\phi = S, Y^2(\phi) \rightarrow M^3(\Gamma)) \]

The pairs of type $(\Theta^4(Y^2, \phi), \Theta^3(\tilde{Y}^2(\phi)))$ are well defined for local pieces and one can glue them in a natural way so as to generate the global objects. What we will do with a bit more details below is to implement this little program of going from local to global.

**Construction of $\Theta^4(Y^2, \phi)$.** For each $\sigma \in \text{Sing}(f) + \text{Sing}(Y^2) \subset Y^2$ we consider, like in figure 1, the undrawable model

\[ K^2(\sigma) \overset{\text{def}}{=} P_1(\sigma) \cup_{\frac{1}{2}L} P_2(\sigma) \subset Y^2. \]

This determines, with a $\tilde{K}^2(\sigma)$ to be defined below, related decompositions

\[ (\phi = S, Y^2(\phi) \rightarrow M^3(\Gamma)) \]

(14) $Y^2 = Y^2$ (non-singular) $\cup \sum_{\sigma} K^2(\sigma)$ and $\tilde{Y}^2(\phi) = Y^2$ (non-singular) $\cup \sum_{\sigma} \tilde{K}^2(\sigma)$.

To $Y^2$ (non-singular), which is *defined* by the formula (14), corresponds a smooth 3-manifold

\[ \Theta^3(Y^2$ (non-singular) $) \subset \Theta^3(\tilde{Y}^2(\phi)). \]

while to $K^2(\sigma)$ corresponds, to begin with, a $\tilde{K}^2(\sigma) \subset Y^2(\phi)$, like in the upper right corner of figure 7, and a $\Theta^3(\tilde{K}^2(\sigma)) \subset \Theta^3(\tilde{Y}^2(\phi))$ which is a copy of $S^1 \times D^2$. [For typographic
simplicity’s sake we have omitted to add a “ϕ” to $\bar{K}^2(\sigma)$ too. There are really two cases for $\bar{K}^2(\sigma)$, only one of which (the I below) is displayed in figure 7, namely

(16) CASE I: $\phi(P_1) = S$, $\phi(P_2) = N$ and

CASE II: $\phi(P_1) = N$, $\phi(P_2) = S$

**Figure 8.** We see here the $\Theta^3(\bar{K}^2(\sigma))$ in the two cases from (16). Inside $\partial \Theta^3(\bar{K}^2(\sigma))$ lives

$\partial K^2(\sigma) = \partial P_1 \cup \partial P_2 \subset \delta(\partial K^2(\sigma)) \subset \partial \Theta^3(\bar{K}^2(\sigma))$, where $\delta(\partial K^2(\sigma))$ is the 2$^d$ regular neighborhood of $\partial K^2(\sigma) \subset \partial \Theta^3(\bar{K}^2(\sigma))$. To the second formula in (14) corresponds also a reconstruction formula for $\Theta^3(\bar{Y}^2(\phi))$, namely

(17) $\Theta^3(\bar{Y}^2(\phi)) = \Theta^3(Y^2 \text{ (non-singular)}) \cup \sum \Theta^3(\bar{K}^2(\sigma))$

Here, with a $\delta(\partial K^2(\sigma)) \supset \partial P_1 \cup \partial P_2 = \partial K^2(\sigma)$ like in figure 8, we have embeddings

(17 - 1) $\partial \Theta^3(Y^2 \text{ (non-singular)} \supset \delta(\partial K^2(\sigma)) \subset \partial \Theta^3(\bar{K}^2(\sigma)))$

along which the pieces in (17) are to be glued.

For each $\sigma$ we consider now a copy of $B^4$

(18) $\Theta^4(K^2(\sigma), \phi) \supset \partial \Theta^4 \supset \Theta^3(\bar{K}^2(\sigma))$.
with the embedding $i$ like in the figure 8 (think of it as representing $\Theta^3 \subset \mathbb{R}^3 \cup \{\infty\} = \partial \Theta^4$).

We finally define

\begin{equation}
\Theta^4(Y^2, \phi) = \Theta^3(Y^2 \text{ (non-singular) }) \times [0, 1] \cup \sum_\sigma \Theta^4(K^2(\sigma), \phi),
\end{equation}

where for each $\sigma$, one should glue

$$
\delta(\partial K^2(\sigma)) \times [0, 1] \subset \partial (\Theta^3(Y^2 \text{ (non-singular) }) \times [0, 1])
$$

coming from the left-hand side of (17–1) with the

$$
\delta(\partial K^2(\sigma)) \times [0, 1] \subset \partial \Theta^4(K^2(\sigma), \phi)
$$

defined by the right-hand side of (17–1), where $\delta(\partial K^2) = \delta(\partial K^2) \times \{0\}$, and which is outgoing with respect to $\Theta^3(K^2(\sigma)) \subset \partial \Theta^4$. \hfill $\square$

In the context of (11) and of lemma 2, we consider now $\Theta^4(Y^2, \phi)$ and $\Theta^4(Y^2, \phi \text{ (induced) })$.

**Lemma 3**

1. In the cases $O(0), O(1), O(2), O(S)$ we have a canonical embedding

\begin{equation}
\Theta^4(Y^2, \phi) \subset \Theta^4(Y^2, \phi))
\end{equation}

which is just a compact smooth Whitehead dilatation.

2. In the coherent $O(3)$ case we have again an embedding (20), this time it is the addition of a handle on index $\lambda = 2$.

3. There is no embedding in the non-coherent $O(3)$ case (see the schematical figure 9).

\begin{figure}[h]
\centering
\begin{tabular}{cc}
\includegraphics[width=0.4\textwidth]{coherent_case} & \includegraphics[width=0.4\textwidth]{non_coherent_case}
\end{tabular}
\caption{In the non-coherent case we see the well known intersection $\mathbb{R}^2 \cap \mathbb{R}^2 \subset \mathbb{R}^4$, standard obstruction in 4-dimensional topology.}
\end{figure}

**Lemma 4** Let $p \geq 1$ and consider

\begin{equation}
\Theta^{n=p+4}(Y^2, \phi) = \Theta^4(Y^2, \phi) \times B^p.
\end{equation}

1. Up to diffeomorphism, (21) is no longer $\phi$-dependant, and we will just denote it by $\Theta^n(Y^2)$. 

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2. With this $\Theta^n$ all the features in the CLAIM B are satisfied.

Proof. In the context of figure 8, we have two distinct embeddings

$$(22) \quad \delta(\partial K^2(\sigma)) \times [0,1] \xrightarrow{\iota_I} \partial \Theta^4(K^2(\sigma), \phi).$$

Here the source is $S^1 \times S^2 - \text{int}D^2 \times [0,1]$ and the target is $S^3$. When we move to $\Theta^{n \geq 5}$, then the corresponding embeddings

$$\delta(\partial K^2(\sigma)) \times [0,1] \times B^p \subset \partial \Theta^n(K^2(\sigma))$$

are now smoothly isotopic. \qed

We consider now singular 3-manifolds $V^3$, of which $M^3(\Gamma)$ and $\tilde{M}^3(\Gamma)$ are examples, with immortal singularities $S$. The singular local model is gotten from figure 1 by changing each $P_1, P_2$ into a thin $P_i \times [-\epsilon, \epsilon]$. These two flat rectangular boxes are then appropriately glued together, so as to change the $\sigma$ (figure 1) into a small square $S$.

**Claim C:** Our claim is that all the little theory above, which has started at (9), extends to $V^3$'s (no $f$ are needed now). This is quite well explained in [4].

With arbitrary abstract desingularizations $\phi$ for $M^3(\Gamma)$ and $\Phi$ for $\tilde{M}^3(\Gamma)$ one can define now $\Theta^n(M^3(\Gamma), \phi), \Theta^n(\tilde{M}^3(\Gamma), \Phi)$ and then the canonical, desingularization-independent ($p \geq 1$)

$$\Theta^{n=p+4}(M^3(\Gamma)) = \Theta^4(M^3(\Gamma), \phi) \times B^p$$
$$\Theta^{n=p+4}(\tilde{M}^3(\Gamma)) = \Theta^4(\tilde{M}^3(\Gamma), \Phi) \times B^p$$

These verify the functorial property (7). \qed

**FINAL COMMENT:** So, as explained in [4], there are really two little theories, one for $2^d$ objects like $Y^2 = X^2 \xrightarrow{f} \tilde{M}^3(\Gamma)$ and another one of $3^d$ objects like $V^3 = \tilde{M}^3(\Gamma)$ itself.

In the present little paper the only necessary bridge between the two is (8-1). In real life one has to go much deeper into the connection between $\Theta^n(Y^2)$ and $\Theta^n(V^3)$.

**References**


